The Application of Invariance Principles to a Radiative Transfer Problem in a Homogeneous Spherical Medium *

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ABSTRACT

The problem of steady-state, radiative transport through a finite, spherically symmetric and uniformly generating medium is considered. The governing equations are applicable to the study of thermal radiation in a heat generating grey gas as well as to the idealization proposed by Cuperman, Engelmann, and Oxenius to study radiation loss in a homogeneous, isothermal spherical plasma due to an optically thick spectral line associated with the de-excitation energy of two-level impurity ions. The analysis is related directly to the study of a one-dimensional problem in a slab of finite optical thickness. This equivalence then permits the direct application of results based on the invariance principles of Ambartsumian and Chandrasekhar. In particular, exact expressions are derived for the source function at the spherical boundary and the radiative flux loss. These results are expressed in terms of tabulated moments of Chandrasekhar's X and Y functions.

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INTRODUCTION

This paper considers the prediction of steady-state radiative transport through a finite and homogeneous spherical medium that is releasing energy uniformly. The main objective is the determination of explicit expressions for the total power loss of the system and the surface value of the source emission function. The analysis is simplified at the outset by establishing a formal equivalence between transport problems in a unidimensional slab and in the spherical region. This equivalence then permits the direct application of special techniques introduced originally by Ambartsumian and Chandrasekhar to treat planar problems.

Since the governing equations have been developed fully elsewhere, a standard formulation is accepted at the outset of the next section. The standard treatises of Chandrasekhar,[1] Kourganoff,[2] and Sobolev[3] provide derivations of the integral transfer equations and, in application, proceed to detailed analysis of radiation in a plane-parallel medium. The equations are sufficiently general to permit different physical interpretations. From one point of view, one may consider the passage of thermal radiation through a heat-generating medium which has a constant volumetric absorption coefficient κ independent of frequency, the so-called grey case, and a known heat-generation rate per unit volume. Thermal transport through spherical enclosures has been calculated numerically by Sparrow, Usiskin, and Hubbard[4] and the present exact results complement that work. Special attention is also directed to the model introduced by Cuperman, Engelmann, and Oxenius[5,6] to characterize the steady-state energy loss of a homogeneous and isothermal plasma due to de-excitation radiation of impurity ions that are assumed to have two energy levels only. In this case, radiation loss is associated with an optically thick spectral line and is calculated

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under the assumption that all other transitions of the ions remain optically thin. The results given below for an extensive range of parameters may be compared directly with the particular numerical cases for a sphere given by Cuperman et al.[5,6] A third physical interpretation involves the isotropic scattering of light in a spherical region.

So far as we are aware the literature of radiation theory contains little exploitation of the close connection between the spherical problem and the planar case. Our initial concern was motivated by intuitive considerations of the similarity between radiative flux for optically thick materials and conductive heat flux where, in the latter case, the formal equivalence for the two geometries is well known (see, e.g., Carslaw and Jaeger,[7] p. 230). After this analysis was completed, it was discovered that the interrelationship has received more attention in the development of neutron transport theory (see, e.g., Davison[8]). Our principal contribution appears in the expression of physically significant quantities, namely, the surface emission function and the power loss, in terms of known physical parameters and moments of the Chandrasekhar-Ambartsumian functions, X and Y. Sobouti[9] has recently supplied tabular values for most of the required moments, and Sobolev,[10] in turn, has given asymptotic expressions for X and Y which aid in the calculation of limiting cases. The needed additions to Sobouti's tables are supplied here. These additional moments, together with expressions obtained through the use of Sobolev's relations, are of some general interest and may prove useful in transfer problems arising in other examples of radiative interchange.

Throughout the analysis, dependence on the radiation frequency is suppressed in the terminology. Whether one interprets the solution in terms of emission associated with a narrow spectral line or of an average over a continuous spectrum is a matter of personal interest and requires no change in notation.

GOVERNING EQUATIONS

Consider a spherical region of radius R. The integral equation governing radiative transfer will be written in terms of the dimensionless variable ρ in a form that differs only in minor details from the terminology of Cuperman et al.,[6] namely,

$$\rho S(\rho) = \rho B(\rho) + \frac{\omega}{2} \int_0^{\kappa R} \rho_1 S(\rho_1)[E_1(|\rho-\rho_1|) - E_1(|\rho+\rho_1|)]\,d\rho_1 \qquad (1)$$

where $E_1(x)$ is the first exponential integral function defined with general index as

$$E_n(x) = \int_1^{\infty} \frac{e^{-xx_1}}{x_1{}^n}\,dx_1 = \int_0^1 e^{-x/\mu}\,\mu^{n-2}\,d\mu \qquad (2)$$

The absorption coefficient κ is averaged over the frequency interval and is assumed to be independent of position, r is distance measured from the center of the sphere, $\rho = \kappa r$, and $S(\rho)$ is, in the language of radiation theory, the source emission function. The interpretation of ω depends on the particular application. In the problem of radiative heat transfer through a nonisothermal, absorbing and heat-generating medium the governing equations are algebraically equivalent if $\omega = 1$. Also $4\pi\kappa B(\rho)$ is power generated per unit volume and the entire frequency range of Planck's function $B_\nu(T)$ is considered from which one gets $S(\rho) = \sigma T^4(r)/\pi$ where $T(r)$ is local temperature. In the model of Cuperman et al.[6] the medium is isothermal and in Eq. (1) $B = (1-\omega)\,B_\nu(T_o)$ where ω is constant and depends on the line absorption coefficient and the excitation process. In the case of isotropic scattering, ω is the spherical indicatrix of scattering and $B(\rho)$ is related either to the incident radiation or an internal source of energy production. Our primary concern here is with the first two applications and attention will be directed to the case $B(\rho)$ a constant.

By an obvious change of variables, Eq. (1) becomes

$$\rho S(\rho) = \rho B(\rho) + \frac{\omega}{2} \int_0^{\kappa R} \rho_1 S(\rho_1) E_1(|\rho-\rho_1|)d\rho_1 - \frac{\omega}{2} \int_0^{-\kappa R} \rho_1 S(-\rho_1) E_1(|\rho-\rho_1|)d\rho_1 \tag{3}$$

The function $S(\rho)$ is defined initially only for $\rho \geq 0$. Extend now its definition so that $S(\rho) = S(-\rho)$ and introduce the transformations

$$\kappa R = \tau_0 , \quad \tau - \tau_0 = \rho , \quad \tau_1 - \tau_0 = \rho_1 , \quad \rho S(\rho)/B = \Omega(\tau) \tag{4}$$

where the source function is made dimensionless through division by the constant value of B. Equation (3) then takes the form

$$\Omega(\tau) = (\tau-\tau_0) + \frac{\omega}{2} \int_0^{2\tau_0} \Omega(\tau_1) E_1(|\tau-\tau_1|)d\tau_1 \tag{5}$$

Equation (5) can be interpreted in terms of radiative transfer through a slab of optical thickness $2\tau_0$ and with a local energy generation rate equal to a linear function of τ. Special methods developed for the latter case thus become directly applicable.

A second integral relation can also be derived for calculating the local energy flux q, that is, the energy per unit time traversing a unit area normal to the radial direction. If $q = q(\rho)$ is expressed as a function of ρ and if a net energy flow in the outward direction has a positive value, the desired relation is

$$\rho^2 q = 2\pi \int_0^{\kappa R} \rho_1 S(\rho_1)[\rho \operatorname{sgn}(\rho-\rho_1)E_2(|\rho-\rho_1|) + E_3(|\rho-\rho_1|)$$

$$- \rho E_2(\rho+\rho_1) - E_3(\rho+\rho_1)]d\rho_1 \tag{6}$$

We shall be especially interested in the total radiation loss from the sphere and for this reason introduce the symbol W, where $W = (\rho^2 q/B\pi)_{\rho=\kappa R}$, to denote this quantity which depends solely on ω and the optical radius κR of the sphere. Evaluating Eq. (6) at $\rho = \kappa R$ one gets

$$W(\omega,\kappa R) = \frac{2}{B} \int_0^{\kappa R} \rho_1 S(\rho_1)[\kappa R E_2(\kappa R-\rho_1) + E_3(\kappa R-\rho_1)$$

$$- \kappa R E_2(\kappa R+\rho_1) - E_3(\kappa R+\rho_1)]d\rho_1 \tag{7a}$$

or

$$W(\omega,\tau_0) = 2 \int_0^{2\tau_0} \Omega(2\tau_0-\tau_1)[\tau_0 E_2(\tau_1) + E_3(\tau_1)]d\tau_1 \tag{7b}$$

Differentiation of Eq. (6) and comparison with Eq. (1) establishes the useful relation

$$\frac{\omega}{4\rho\pi} \frac{d}{d\rho} (\rho^2 q) = B\rho - (1-\omega)\rho S(\rho) \tag{8a}$$

and this reduces, when B is constant, to

$$\frac{\omega}{4(\tau-\tau_0)} \frac{d}{d\tau}\left(\frac{\rho^2 q}{\pi B}\right) = (\tau-\tau_0) - (1-\omega)\Omega(\tau) \tag{8b}$$

In the next section a résumé is given of methods whereby surface values of $\Omega(\tau)$ are made available along with the dimensionless radiation loss $W(\omega,\tau_o)$. The mathematical problem is one of inverting the integral Eq. (5) and this directs attention to the nature of the resolvent kernel of the equation. Manipulative details have been avoided since they are at times lengthy and are closely related to the basic calculations of Chandrasekhar[1] and Sobolev.[3] We have chosen, instead, to concentrate on explicit formulas and expansion procedures that aid in their evaluation.

APPLICATION OF INVARIANCE PRINCIPLES

Equation (5) is a Fredholm integral equation with symmetric kernel. It has the general form

$$F(\tau) = G(\tau) + \frac{\omega}{2}\int_0^{\tau_a} F(\tau_1)E_1(|\tau-\tau_1|)d\tau_1 \tag{9a}$$

and, after inversion, may be written

$$F(\tau) = G(\tau) + \int_0^{\tau_a} G(\tau_1)L(\tau,\tau_1;\tau_a)d\tau_1 \tag{9b}$$

where $L(\tau,\tau_1;\tau_a)$ is the resolvent kernel which is a symmetric function of τ and τ_1. A grasp on the analytic nature of L is achieved through consideration of differential invariants for the specific kernel in hand. This approach, which arose initially from the physical concept that for infinite optical thickness surface conditions are not modified by inclusion of an additional layer of limited depth, can be extended formally to yield

$$\frac{\partial L}{\partial \tau} + \frac{\partial L}{\partial \tau_1} = \Phi(\tau_1)\Phi(\tau) - \Phi(\tau_a-\tau_1)\Phi(\tau_a-\tau) \tag{10a}$$

where

$$\Phi(\tau) = L(0,\tau;\tau_a) \tag{10b}$$

When $\tau_1 > \tau$, Eq. (10a) yields

$$L(\tau_1,\tau;\tau_a) = \Phi(\tau_1-\tau)$$

$$+ \int_0^{\tau} [\Phi(t)\Phi(t+\tau_1-\tau) - \Phi(\tau_a-t)\Phi(\tau_a-t-\tau_1+\tau)]dt \tag{11}$$

The symmetric kernel can thus be calculated in terms of the unidimensional function $\Phi(\tau)$ where $\Phi(\tau)$ satisfies the integral equation

$$\Phi(\tau) = \frac{\omega}{2} E_1(\tau) + \frac{\omega}{2} \int_0^{\tau_a} \Phi(\tau_1)E_1(|\tau-\tau_1|)d\tau_1 \tag{12}$$

If Eqs. (9a) and (9b) are specialized, one gets the relations

$$F(\tau,\mu) = e^{-\tau/\mu} + \frac{\omega}{2} \int_0^{\tau_a} F(\tau_1,\mu)E_1(|\tau-\tau_1|)d\tau_1 \tag{13}$$

$$F(\tau,\mu) = e^{-\tau/\mu} + \int_0^{\tau_a} e^{-\tau_1/\mu} L(\tau,\tau_1;\tau_a)d\tau_1 \tag{14}$$

The values of $F(\tau,\mu)$ at $\tau = 0$ and $\tau = \tau_a$ are the Chandrasekhar-Ambartsumian functions. Denoting these values, respectively, by $X(\mu,\tau_a)$ and $Y(\mu,\tau_a)$, we get from Eq. (14)

$$X(\mu,\tau_a) = 1 + \int_0^{\tau_a} \Phi(\tau_1)e^{-\tau_1/\mu} d\tau_1 \tag{15}$$

$$Y(\mu,\tau_a) = e^{-\tau_a/\mu} + \int_0^{\tau_a} \Phi(\tau_a-\tau_1)e^{-\tau_1/\mu}\, d\tau_1 \tag{16}$$

In the Russian literature these functions are usually denoted as $\varphi(\mu)$ and $\psi(\mu)$ and referred to as Ambartsumian functions. They actually depend on the three parameters μ, τ_a, and ω. It suffices here to state that they have been computed for a wide range of the parameters and that Sobouti[9] has also provided their moments α_n, β_n, $n = 0, 1, 2$ where

$$\alpha_n(\tau_a) = \int_0^1 X(\mu,\tau_a)\mu^n\, d\mu \ . \tag{17}$$

$$\beta_n(\tau_a) = \int_0^1 Y(\mu,\tau_a)\mu^n\, d\mu \tag{18}$$

The values of these moments will determine the quantities we seek.

From comparison of Eqs. (5), (9), and (11) we have, after setting $\tau_a = 2\tau_o$,

$$\begin{aligned}\Omega(2\tau_o) &= -\Omega(0) = \tau_o S(\tau_o)/B \\ &= \tau_o + \int_0^{2\tau_o} (\tau_o-\tau_1)\Phi(\tau_1;2\tau_o)d\tau_1 \\ &= -\tau_o\Psi(2\tau_o;2\tau_o) + \int_0^{2\tau_o} \Psi(\tau_1;2\tau_o)d\tau_1\end{aligned} \tag{19}$$

where

$$\Psi(\tau;2\tau_o) = 1 + \int_0^{\tau} \Phi(\tau_1;2\tau_o)d\tau_1 \tag{20}$$

Detailed evaluation of these expressions, starting with Eq. (12), leads to the following results

$$\Psi(2\tau_o;2\tau_o) = \frac{1}{1 - \frac{\omega}{2}(\alpha_o-\beta_o)} \tag{21}$$

$$\Omega(2\tau_o) = \frac{\tau_o\left[1 - \frac{\omega}{2}(\alpha_o-\beta_o)\right] - \frac{\omega}{2}(\alpha_1-\beta_1)}{(1-\omega)} \tag{22}$$

In all relations $\alpha_n = \alpha_n(\tau_a) = \alpha_n(2\tau_o)$, $\beta_n = \beta_n(\tau_a) = \beta_n(2\tau_o)$.

Chandrasekhar[1] and Sobolev[3] have derived fundamental identities satisfied by the moment functions. The following relations are useful as checks of numerical accuracy and also in algebraic manipulation of formulas

$$\left.\begin{aligned} \alpha_o &= 1 + \frac{1}{4}\omega(\alpha_o^2-\beta_o^2) \\ \alpha_1 - \beta_1 &= \tau_a\left[1 - \frac{1}{2}(\alpha_o-\beta_o)\right], \qquad \omega = 1 \\ 2\alpha_2 - \frac{2}{3} &= \omega\left[(\alpha_2\alpha_o-\beta_2\beta_o) - \frac{1}{2}(\alpha_1^2-\beta_1^2)\right] \end{aligned}\right\} \tag{23}$$

Equation (22) provides the exact prediction of the surface values of the source emission function. It reduces to an indeterminate form in the special case $\omega = 1$. The desired expression then becomes

$$\Omega(2\tau_o) = \frac{1}{2}\tau_o^2(\alpha_1-\beta_1) + \frac{3}{2}\tau_o(\alpha_2-\beta_2) + \frac{3}{2}(\alpha_3-\beta_3), \qquad \omega = 1 \tag{24}$$

The evaluation of radiation loss, as expressed by Eq. (7b), depends upon a knowledge of the variation of $\Omega(\tau)$. Since the resolvent kernel is expressed in terms of $\Phi(\tau)$ by Eq. (11), and $\Phi(\tau)$ is related to $\Psi(\tau)$

by Eq. (20), the inversion of Eq. (9b), with $F(\tau)$ replaced by $\Omega(\tau)$ and $G(\tau)$ replaced by $\tau - \tau_o$, can be manipulated algebraically to give

$$\Omega(\tau) = -\tau\Psi^2(2\tau_o;2\tau_o) + \Psi(2\tau_o;2\tau_o)\int_0^\tau [\Psi(\tau_1;2\tau_o) + \Psi(2\tau_o-\tau_1;2\tau_o)]d\tau_1$$

$$+ [\Psi(\tau;2\tau_o) - \Psi(2\tau_o-\tau;2\tau_o) + \Psi(2\tau_o;2\tau_o)]$$

$$\cdot \left[\tau_o\Psi(2\tau_o;2\tau_o) - \int_0^{2\tau_o} \Psi(\tau_1;2\tau_o)d\tau_1\right] \qquad (25)$$

Substitution into Eq. (7b) permits one to arrive at a prediction in terms of known functions. After algebraic simplification the formula becomes

$$\left.\begin{aligned} W(\omega,\tau_o) &= 2\Omega(2\tau_o)[(\alpha_2+\beta_2) + \tau_o(\alpha_1+\beta_1)] \\ &\quad - 2\Psi(2\tau_o;2\tau_o)[(\alpha_3-\beta_3) + \tau_o(\alpha_2-\beta_2)] , \qquad \omega \neq 1 \\ W(1,\tau_o) &= \frac{4}{3}\tau_o^3 \end{aligned}\right\} \qquad (26)$$

When $\omega = 1$ the radiation loss function W can be derived directly through physical considerations of radiative heat transfer. If q is energy flux at the surface of the sphere, a balance between total rate of loss at the surface and total rate of energy production within the sphere yields the relation $4\pi R^2 q = (4\pi R^3/3)[4\pi\kappa B]$, where the bracketed term, as noted previously, is rate of energy production per unit volume. The equality reduces immediately to $W(1,\tau_o) = 4\tau_o^3/3$. Integration of Eq. (8b) from $\tau = \tau_o$ to $\tau = 2\tau_o$ leads to the same conclusion when $\omega = 1$.

Equations (22), (24), and (26) comprise the specific results of this section. Actual predictions now follow by means of simple arithmetic operations through use of the moment functions of X and Y. Sobouti's[9] tables give α_n and β_n for $n = 0, 1, 2,$ and $2\tau_0 \leq 3$. The accompanying Table I gives additional values of α_3 and β_3 computed from Sobouti's[9] table of X and Y functions. Graphical results will be given in the next section.

For optical thicknesses of increasing magnitude it is convenient to resort to asymptotic relations given by Sobolev.[10] When $\omega = 1$, Sobolev's results are

$$X(\mu,\tau_a) \sim H(\mu) - \mu H(\mu)/(\tau_a+\gamma) \tag{27}$$

$$Y(\mu,\tau_a) \sim \mu H(\mu)/(\tau_a+\gamma) \tag{28}$$

Here, $H(\mu)$ is Chandrasekhar's[1] H function which corresponds to the planar problem for a semi-infinite medium. The constant γ is twice the ratio of the second and first moments of $H(\mu)$. Denoting the nth moment of $H(\mu)$ by $\bar{\alpha}_n$ where $\bar{\alpha}_n = \alpha_n(\infty)$, the following values apply when $\omega = 1$

$$\left.\begin{array}{ll} \bar{\alpha}_0 = \alpha_0(\infty) = 2 & \bar{\alpha}_1 = \alpha_1(\infty) = 2/\sqrt{3} \\ \bar{\alpha}_2 = \alpha_2(\infty) = 0.82035 & \bar{\alpha}_3 = \alpha_3(\infty) = 0.63782 \\ \bar{\alpha}_4 = \alpha_4(\infty) = 0.52223 & \gamma = 1.42089 \end{array}\right\} \tag{29}$$

The values for $n = 3, 4$ were calculated from the tabulated values of $H(\mu)$ given by Stibbs and Weir.[11] Previously given values for $n = 0, 1, 2$ were recalculated as a check on the numerical accuracy. Differences appeared only in the fifth decimal place.

From Eqs. (27) and (28) the asymptotic moment relations for $\omega = 1$ are

$$\left.\begin{aligned} \alpha_n(\tau_a) &\sim \bar{\alpha}_n - \frac{\bar{\alpha}_{n+1}}{\tau_a + \gamma} \\ \beta_n(\tau_a) &\sim \frac{\bar{\alpha}_{n+1}}{\tau_a + \gamma} \end{aligned}\right\} \tag{30}$$

These relations, together with Eqs. (24) and (26), provide the following predictions at $\omega = 1$

$$\frac{\Omega(2\tau_o)}{\tau_o} \sim \frac{1}{2}\tau_o\bar{\alpha}_1 + \frac{3}{2}\bar{\alpha}_2 + \frac{3\bar{\alpha}_3}{2\tau_o} - \frac{1}{2\tau_o + \gamma}\left(\tau_o\bar{\alpha}_2 + 3\bar{\alpha}_3 + \frac{3\bar{\alpha}_4}{\tau_o}\right) \tag{31}$$

$$W(1,\tau_o) \sim (\gamma+2\tau_o)\left[\frac{2}{\sqrt{3}}\Omega(2\tau_o) - \sqrt{3}(\bar{\alpha}_3+\tau_o\bar{\alpha}_2)\right] + 2\sqrt{3}(\bar{\alpha}_4+\tau_o\bar{\alpha}_3) = \frac{4}{3}\tau_o^3 \tag{32}$$

Values calculated from Eqs. (31) and (32) may be compared directly with the tabulations of surface source strength and energy loss given by Cuperman et al.[6] At $\tau_o = 10$ agreement is excellent for their most precise calculations of the emission function. Their approximate predictions of energy loss are, however, more accurate than were the detailed calculations.

It remains to estimate the range of τ_a for which the asymptotic expressions can be used safely. A measure of this range follows after comparison of the predictions in Eqs. (30) and Sobouti's numerical evaluation of the moments. Sobouti's tables for $n = 0, 1, 2$ stop at $\tau_a = 3$ and at this end point the differences between the predictions of Eqs. (30) and the tabulated values differ at worst in the fourth decimal place. Equations (31) and (32) should therefore yield about four figure accuracy for $\tau_o \geq 1.5$.

When $\omega \neq 1$, Sobolev's[10] asymptotic relations are less simple. In the present notation, one has

$$X(\mu,\tau_a) \sim H(\mu) - C \frac{\mu}{1-k\mu} H(\mu)e^{-2k\tau_a} \tag{33}$$

$$Y(\mu,\tau_a) \sim C_1 \frac{\mu}{1-k\mu} H(\mu)e^{-k\tau_a} \tag{34}$$

The parameter k depends on ω and is a root of the characteristic equation

$$\frac{\omega}{2} \int_o^1 \frac{H(\mu)d\mu}{1-k\mu} = \frac{\omega}{2k} \ln \frac{1+k}{1-k} = 1 \tag{35}$$

Its values are given by Chandrasekhar[1] (see p. 19). The parameters C and C_1 satisfy the relations

$$C \int_o^1 \frac{H(\mu)\mu \, d\mu}{(1-k\mu)^2} = C_1 \int_o^1 \frac{H(\mu)\mu \, d\mu}{1-k^2\mu^2} \tag{36}$$

$$C_1 \int_o^1 \frac{H(\mu)\mu \, d\mu}{(1-k\mu)^2} = 2k \int_o^1 \frac{H(\mu)\mu \, d\mu}{1-k^2\mu^2} \tag{37}$$

Taking moments of Eqs. (33) and (34), one has

$$\alpha_n(\tau_a) \sim \bar{\alpha}_n - Ce^{-2k\tau_a} \int_o^1 \frac{\mu^{n+1}H(\mu)}{1-k\mu} d\mu \tag{38}$$

$$\beta_n(\tau_a) \sim C_1e^{-k\tau_a} \int_o^1 \frac{\mu^{n+1}H(\mu)}{1-k\mu} d\mu \tag{39}$$

Thus, by virtue of Eq. (35)

$$\alpha_n(\tau_a) \sim \bar{\alpha}_n - C\left(\frac{e^{-2k\tau_a}}{k^{n+1}}\right)\left(\frac{2}{\omega} - \sum_{m=0}^{n} k^m \bar{\alpha}_m\right) \tag{40}$$

$$\beta_n(\tau_a) \sim C_1\left(\frac{e^{-k\tau_a}}{k^{n+1}}\right)\left(\frac{2}{\omega} - \sum_{m=0}^{n} k^m \bar{\alpha}_m\right) \tag{41}$$

For the present purposes, a more convenient approximation follows by using tabulated values of $\alpha_n(\tau_a)$ and $\beta_n(\tau_a)$ for all $\tau_a \leq 3$ and extrapolating for greater thicknesses by assuming the tabulations agree with the asymptotic predictions at $\tau_a = 3$. This modification yields

$$\alpha_n(\tau_a) \sim \bar{\alpha}_n - [\bar{\alpha}_n - \alpha_n(3)]e^{-2k(\tau_a-3)} \tag{42}$$

$$\beta_n(\tau_a) \sim \beta_n(3)e^{-k(\tau_a-3)} \tag{43}$$

So far as is known, the variation of the parameters C and C_1 with ω is not available in the literature. Precise calculation over the full range of ω and k demands a considerable degree of accuracy in $H(\mu)$. In the results to follow we have used Eqs. (42) and (43) throughout.

For small values of optical thickness, Chandrasekhar[1] (p. 204, et seq.) has given the appropriate expansions for the X and Y functions and their moments. The simplest forms of these expressions are

$$\left.\begin{aligned}
X(\mu,\tau_a) &\approx 1 + \Delta(\tau_a)\mu(1-e^{-\tau_a/\mu}) \\
Y(\mu,\tau_a) &\approx e^{-\tau_a/\mu} + \Delta(\tau_a)\mu(1-e^{\tau_a/\mu}) \\
\alpha_n(\tau_a) &\approx \frac{1}{n+1} + \Delta(\tau_a)\left[\frac{1}{n+2} - E_{n+3}(\tau_a)\right] \\
\beta_n(\tau_a) &\approx E_{n+2}(\tau_a) + \Delta(\tau_a)\left[\frac{1}{n+2} - E_{n+3}(\tau_a)\right]
\end{aligned}\right\} \tag{44}$$

where

$$\Delta(\tau_a) = \frac{1 - \frac{1}{2}\,\omega[1+E_2(\tau_a)] - \dfrac{(1-\omega)}{1-\frac{1}{2}\,\omega[1-E_2(\tau_a)]}}{\omega\left[\frac{1}{2} - E_3(\tau_a)\right]}$$

The range of validity of the approximations is small and more accurate representations are available. If accuracy beyond three decimal places is desired, use of Eqs. (44) should be limited to $\tau_a < 0.1$.

RESULTS

Figure 1 shows graphically the dependence of the source function at the surface, $\Omega(2\tau_o)/\tau_o = S(\tau_o)/B$, on ω and the optical radius of the sphere. The curves represent the predictions of Eqs. (22) and (24) and the points used in plotting the curves were calculated for those values of ω listed in Table I. From Eq. (22) it follows that when $\omega = 0$, $\Omega(2\tau_o)/\tau_o$ is independent of τ_o. As ω nears 1 it is apparent that increasing demands are made on the accuracy of $(\alpha_o-\beta_o)$ and $(\alpha_1-\beta_1)$. The limiting case, $\omega = 1$, is given exactly by Eq. (24), however, and a consistent, monotonic variation of the graphs followed from the tabulated moments except for $\omega = 0.95$, Fig. 1

$\tau_0 \leq 2$. In this region the curves were faired to conform with the known end value. Isolated comparisons with published predictions[4,6] show good agreement. From Sobouti's tables the following results were determined for $\omega = 1$

τ_0	0.5	1.0	1.5	2.0	3.0	5.0
$\Omega(2\tau_0)/\tau_0$	1.256	1.520	1.787	2.064	2.622	3.755

At this value of ω, Eq. (31) gives an approximate expression for the source function when $\tau_0 \gg 1$; when $\tau_0 \ll 1$ Eqs. (24) and (44) yield $\Omega(2\tau_0)/\tau_0 \approx 1 + O(\tau_0^3)$.

Figure 2 shows the variation of the radiation-loss function as calculated from Eqs. (26). An analytic contrast appears in the calculation of the source function and the loss function at extreme values of ω. Thus, the former is fixed at $\omega = 0$ and has a more complex behavior at $\omega = 1$; the latter is determined simply at $\omega = 1$ and is subject to more careful consideration at $\omega = 0$. Since, however, $\Omega(\tau) = \tau - \tau_0$ when $\omega = 0$, this variation may be used together with Eq. (7b) to get Fig. 2

$$W(0,\tau_0) = \frac{1}{2}\,(1+2\tau_0)e^{-2\tau_0} + \tau_0^2 - \frac{1}{2} \tag{45}$$

When $\tau_0 \ll 1$, one has $W(0,\tau_0) \approx \frac{4}{3}\,\tau_0^3 + O(\tau_0^4)$ and when $\tau_0 \gg 1$, $W(0,\tau_0) \sim \tau_0^2 + O(1)$. Again, the tabular values of the moment functions gave smooth, monotonic functions of ω and τ_0 throughout the range of calculation except for $\omega = 0.95$, $\tau_0 \leq 2$.

The previous work has been based entirely on the use of functions and moments derived originally for use in astrophysical problems. In these concluding paragraphs the solution to Eq. (1), or (5), for $\omega = 1$ is related to other numerical results that have appeared within the past few years in the engineering literature. In the latter case, the particular problem of radiative heat transfer between heated parallel walls and through a grey medium that may be generating energy has been considered (see, e.g., Usiskin and Sparrow,[12] Viskanta and Grosh,[13] and Howell and Perlmutter[14]). In the most general form of the problem the walls have known emission coefficients but it is possible to reduce the analysis to black-wall emission. These details will not be repeated here. It is to be remarked merely that the general solution may be expressed as a linear combination of solutions of the uncoupled integral equations

$$\varphi(\tau) = \frac{1}{2} E_2(\tau_a - \tau) + \frac{1}{2} \int_0^{\tau_a} \varphi(\tau_1) E_1(|\tau - \tau_1|) d\tau_1 \qquad (46)$$

$$\varphi_s(\tau) = \frac{1}{4} + \frac{1}{2} \int_0^{\tau_a} \varphi_s(\tau_1) E_1(|\tau - \tau_1|) d\tau_1 \qquad (47)$$

where $\tau = \kappa x$ and x is the distance measured from one of the walls. Optical distance between the walls is τ_a. The dependent functions are representations of dimensionless temperature functions

$$\varphi(\tau) = \left[\frac{\sigma T^4(x) - \sigma T_{W1}^4}{\sigma T_{W2}^4 - \sigma T_{W1}^4} \right]_{B=0} \qquad (48)$$

$$\varphi_s(\tau) = \left[\frac{\sigma T^4(x) - \sigma T_{W1}^4}{4\pi B} \right]_{T_{W1}=T_{W2}} \qquad (49)$$

where $T(x)$ is temperature distribution through the medium; T_{w1} and T_{w2} are, respectively, the two wall temperatures and σ is the Stefan-Boltzmann constant. In both instances the walls emit as black bodies. Equation (48) is the solution for $B = 0$ and Eq. (49) the solution for $T_{w1} = T_{w2}$. Calculations of these functions have been carried out ab initio and the results may be found in the cited references and elsewhere. In most cases the solutions are presented graphically and can be read to, at most, two or three significant figures.

Known solutions to Eqs. (46) and (47) can be transformed without excessive labor into the solution of Eq. (5). Starting with Eq. (47), a new function $\tilde{\Omega}(\tau)$ is introduced where

$$\tilde{\Omega}(\tau) = 4 \int_{\tau_a/2}^{\tau} \varphi_s(\tau_1)d\tau_1 \tag{50}$$

Equations (50) and (47) yield

$$\tilde{\Omega}(\tau) = \left(\tau - \frac{1}{2}\tau_a\right) + \frac{1}{2}\tilde{\Omega}(\tau_a)[E_2(\tau_a-\tau)-E_2(\tau)] + \frac{1}{2}\int_0^{\tau_a} \tilde{\Omega}(\tau_1)E_1(|\tau-\tau_1|)d\tau_1 \tag{51}$$

The solution $\Omega(\tau)$ of Eq. (5) for $\omega = 1$ can now be expressed as a linear combination of $\tilde{\Omega}(\tau)$ and $\varphi(\tau)$. Setting $\tau_a = 2\tau_o$ one has

$$\Omega(\tau) = \tilde{\Omega}(\tau) + \tilde{\Omega}(2\tau_o)[1-2\varphi(\tau)] \tag{52}$$

Equation (52) relates the variation of the emission function $\Omega(\tau)$ to $\varphi(\tau)$ and $\tilde{\Omega}(\tau)$, the latter following from an additional running integration of the known function $\varphi_s(\tau)$.

Figure (3) shows the source function $S(\rho)/B = \Omega(\tau)/(\tau-\tau_o)$ calculated from Eq. (52) by using unpublished numerical solutions of Eqs. (46) and (47) computed by the present writers. Fig. 3

FOOTNOTES

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FIGURE LEGENDS

Fig. 1.- Dimensionless source function at surface of sphere, showing dependence on optical radius and the parameter ω.

Fig. 2.- Dimensionless radiation-loss function for sphere, showing dependence on optical radius and the parameter ω.

Fig. 3.- Variation of source function within sphere for selected values of optical radius and $\omega = 1$.

TABLE I. THIRD-ORDER MOMENTS OF CHANDRASEKHAR'S X AND Y FUNCTIONS

ω	$\tau_1 = 0.1$		$\tau_1 = 0.2$		$\tau_1 = 0.5$		$\tau_1 = 1.0$		$\tau_1 = 2.0$		$\tau_1 = 3.0$	
	α_3	β_3	α_3	β_3	α_3	β_3	α_3	β_3	α_3	β_3	α_3	β_3
0.10	0.2533	0.2223	0.2549	0.1969	0.2571	0.1368	0.2582	0.0753	0.2585	0.0235	0.2585	0.0076
0.20	0.2568	0.2256	0.2601	0.2018	0.2649	0.1433	0.2673	0.0809	0.2680	0.0262	0.2681	0.0087
0.30	0.2603	0.2291	0.2656	0.2070	0.2734	0.1504	0.2775	0.0874	0.2789	0.0296	0.2791	0.0102
0.40	0.2640	0.2328	0.2714	0.2125	0.2827	0.1583	0.2891	0.0951	0.2916	0.0340	0.2919	0.0122
0.50	0.2679	0.2365	0.2775	0.2183	0.2930	0.1672	0.3025	0.1043	0.3065	0.0397	0.3070	0.0150
0.60	0.2718	0.2404	0.2840	0.2246	0.3045	0.1772	0.3181	0.1155	0.3247	0.0474	0.3256	0.0193
0.70	0.2759	0.2444	0.2909	0.2312	0.3174	0.1885	0.3367	0.1292	0.3473	0.0583	0.3492	0.0259
0.80	0.2802	0.2486	0.2983	0.2382	0.3319	0.2014	0.3592	0.1466	0.3772	0.0747	0.3813	0.0373
0.85	0.2823	0.2507	0.3021	0.2419	0.3400	0.2086	0.3723	0.1570	0.3961	0.0861	0.4026	0.0466
0.90	0.2846	0.2529	0.3061	0.2457	0.3485	0.2163	0.3872	0.1691	0.4192	0.1011	0.4296	0.0600
0.95	0.2868	0.2551	0.3102	0.2497	0.3578	0.2247	0.4041	0.1830	0.4479	0.1212	0.4661	0.0811
1.00	0.2891	0.2574	0.3145	0.2538	0.3677	0.2337	0.4234	0.1993	0.4855	0.1490	0.5197	0.1174

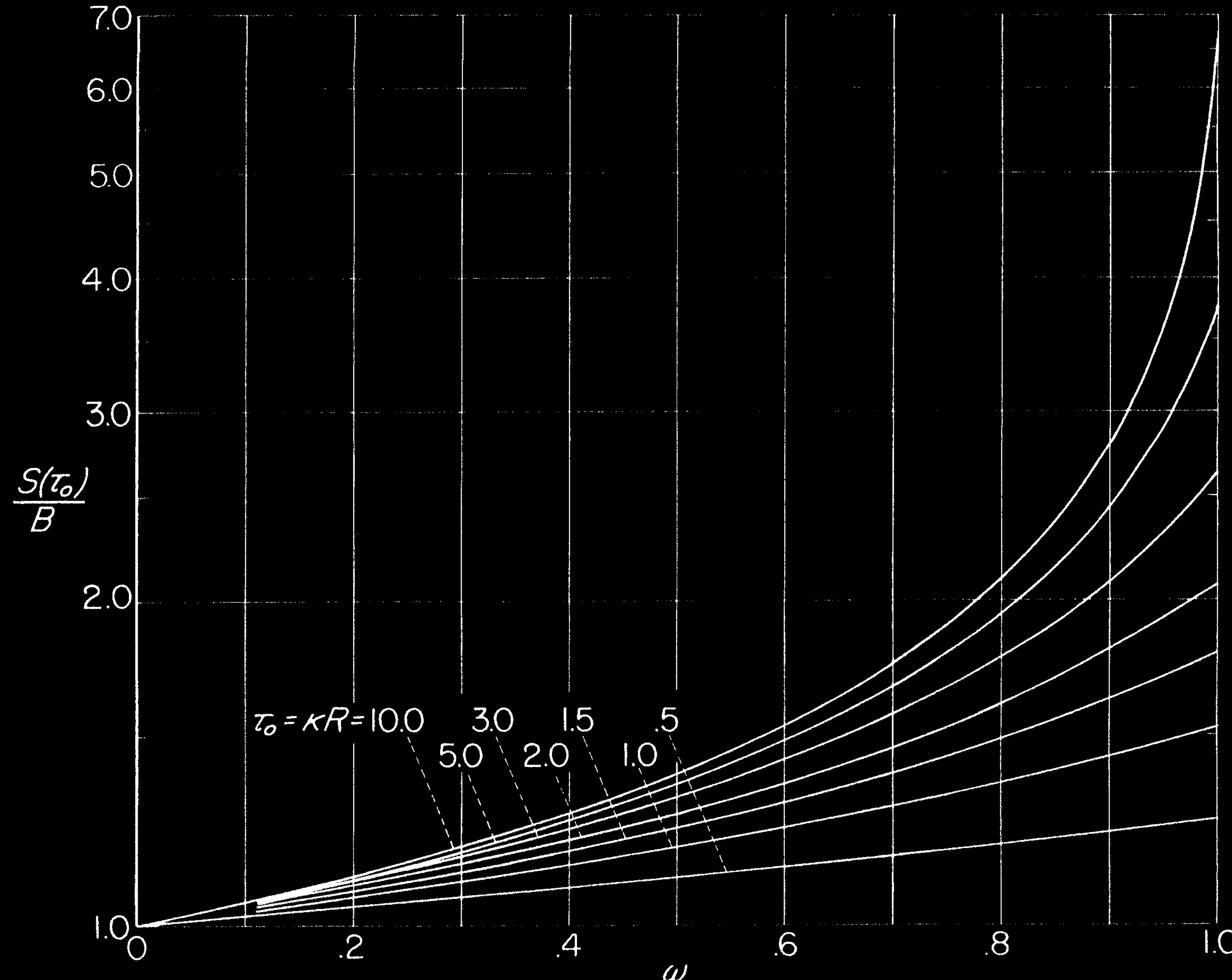

Fig. 1.- Dimensionless source function at surface of sphere, showing dependence on optical radius and the parameter ω.

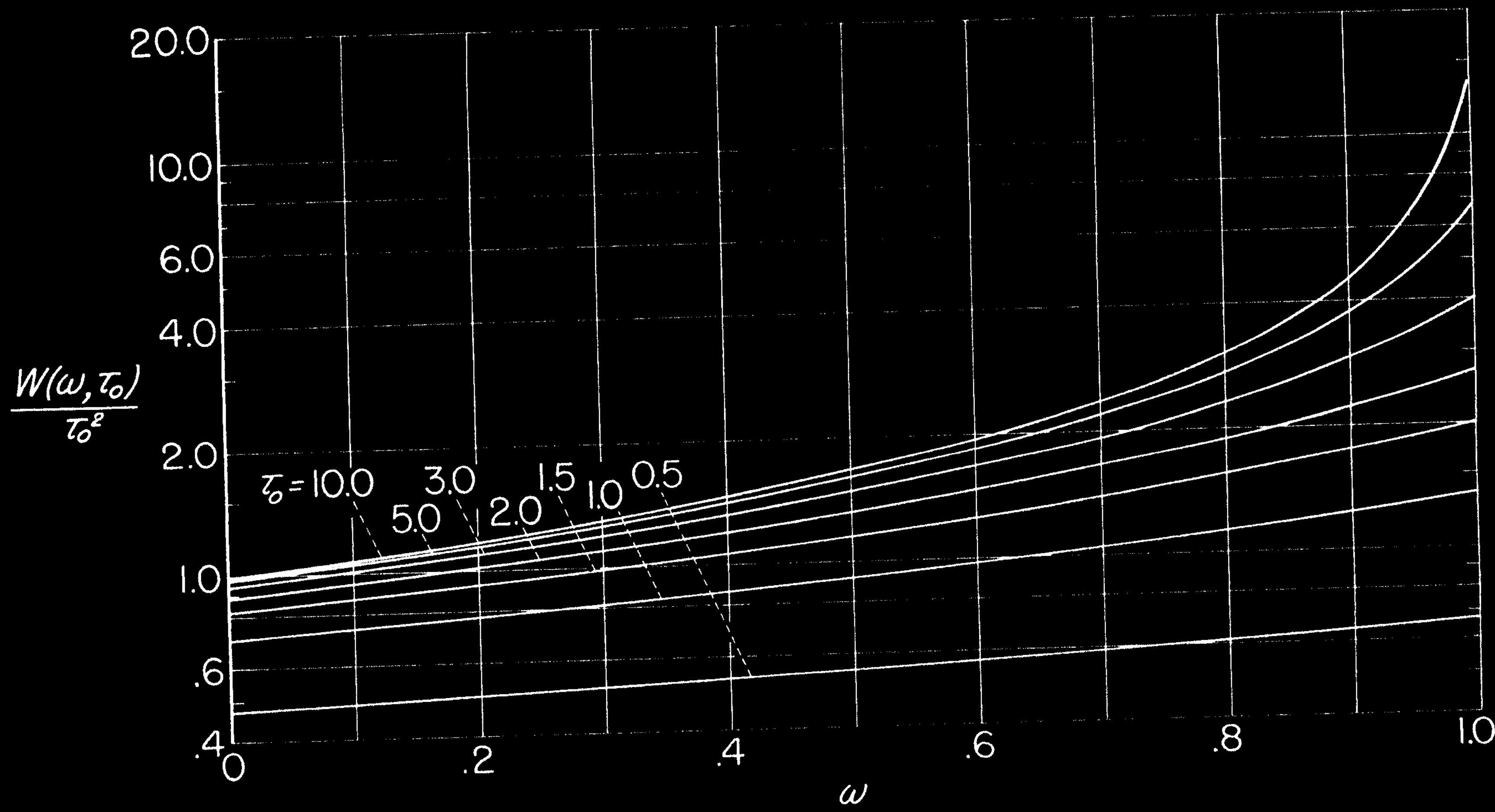

Fig. 2.- Dimensionless radiation-loss function for sphere, showing dependence on optical radius and the parameter ω.

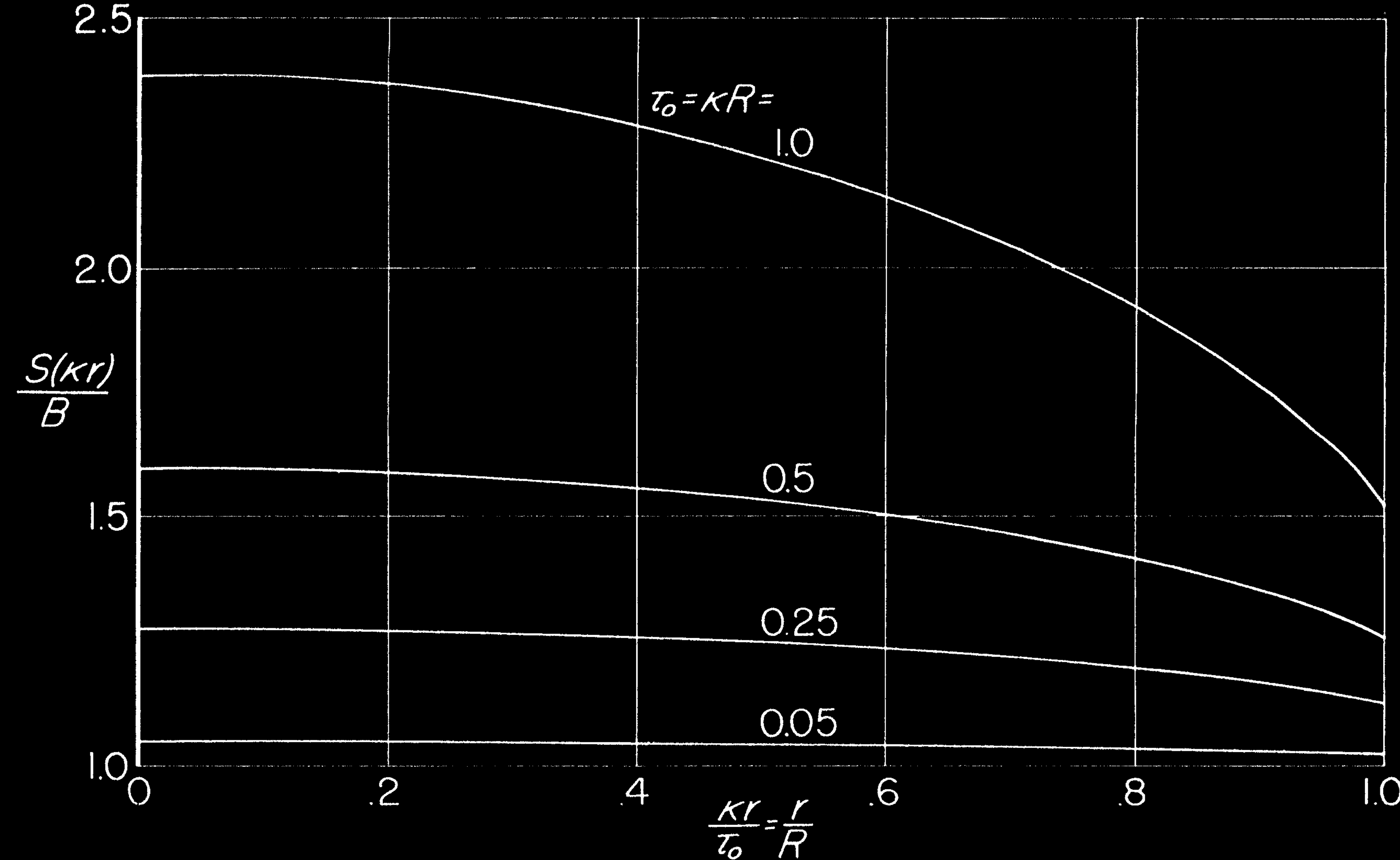

Fig. 3.- Variation of source function within sphere for selected values of optical radius and $\omega = 1$.